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On remarkable properties of invariance superalgebras for the harmonic oscillator

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Abstract. Starting from the orthosymplectic superalgebra osp(3/4), we rigorously explain the correspondence between two superalgebras recently proposed in this journal as invariance superalgebras of the one-dimensional quantum harmonic oscillator, i.e. osp(2/2) $\oplus$ sh(2/2) and osp(3/2). The study of the corresponding root systems as well as their properties with respect to their even or odd characters gives us remarkable properties of the enclosed subsuperalgebras of osp(3/4). Such properties and correspondences can be extended to the n-dimensional context.

The *largest* invariance superalgebra for the n-dimensional quantum harmonic oscillator has recently been determined (Beckers *et al* 1988a). It is the semi-direct sum of the (simple) orthosymplectic Lie superalgebra osp($2n/2n$) determined by de Crombrugghe and Rittenberg (1983) and the (non-simple) Heisenberg superalgebra sh($2n, 2n$) first introduced by Beckers and Hussin (1986). Such an invariance superalgebra corresponds to kinematical (Niederer 1973) as well as to dynamical (Wybourne 1974) (super)symmetries.

Another invariance superalgebra for the n-dimensional harmonic oscillator has also been pointed out in the journal by Englefield (1988), i.e. the (simple) orthosymplectic superalgebra osp($3, 2n$) as the generalisation of the Van der Jeugt structure (Van der Jeugt 1984) for the one-dimensional case.

Due to the inclusion

$$[\mathrm{osp}(2n/2n) \oplus \mathrm{sh}(2n/2n)] \supset \mathrm{osp}(3, 2n) \qquad \forall n \geqslant 1 \tag{1}$$

the link between these two proposals has just been published (Beckers *et al* 1988b). It is realised through the correspondence

$$[\mathrm{osp}(2/2n) \oplus \mathrm{sh}(2/2n)] \leftrightarrow \mathrm{osp}(3, 2n). \tag{2}$$

We notice that these last two superalgebras have the same number of generators (up to the identity operator of the central extension of the Heisenberg structure). The superalgebra osp($2/2n$) $\oplus$ sh($2/2n$) refers to *only* two fermionic degrees of freedom while to $2n$ bosonic ones. The argument proposed in order to get the correspondence (2) is a 'trick' based on the fundamental role played by the Heisenberg generators (Beckers and Hussin 1986). In fact, by inverting the even and odd characters of the non-trivial Heisenberg generators, Beckers *et al* (1988b) have reconstructed the Lie subalgebra contents of osp(3, $2n$), i.e. so(3) and sp($2n$), starting from osp($2/2n$)$\oplus$ sh($2/2n$). Such a correspondence is not an isomorphism but the argument can now be explained rigorously. This is the main purpose of this paper where we mainly address ourselves for simplicity to the one-dimensional context, i.e. to the correspondence

$$\mathrm{osp}(2/2)\oplus\mathrm{sh}(2/2)\leftrightarrow\mathrm{osp}(3/2) \tag{3}$$

dealing with 12-dimensional superalgebras.

Let us recall some notation and conventions (Cornwell 1989). Starting from the complex associative superalgebra $M(p/q;\mathbb{C})$ easily handled through the $(p+q)\times(p+q)$ matrices e_{kl} defined by

$$(e_{kl})_{ij}=\delta_{ki}\delta_{lj} \qquad i,j,k,l=1,2,\ldots,p+q \tag{4}$$

the complex orthosymplectic Lie superalgebra osp(p/q) is the set of matrices M of $M(p/q;\mathbb{C})$ with $p\geqslant 1$ and with q positive and *even* such that

$$M^{\mathrm{st}}K+(-1)^{\deg M}KM=0. \tag{5}$$

Here M^{st} indicates the supertranspose of M, deg M denotes the degree of M and the metric is realised through

$$K=\begin{pmatrix}G_p & 0\\ 0 & J_q\end{pmatrix} \qquad G_p=(1_p) \qquad J_q=\begin{pmatrix}0 & 1_{q/2}\\ -1_{q/2} & 0\end{pmatrix}. \tag{6}$$

In the following analysis instead of taking $G_p=1_p$, it will be convenient to take

$$G_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad G_3=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{7}$$

for $p=2$ and 3 respectively so that we get isomorphic superalgebras to osp($2/q$) and osp($3/q$). In particular the Lie superalgebras osp(3/4), osp(3/2) and osp(2/2) have now to be considered for our purpose: they are all simple and in the notation of Kac (1977a, b) they are denoted by B(1/2), B(1/1) and C(2) respectively.

The 25-dimensional structure of osp(3/4) has 13 as even dimension and 12 as odd dimension; its rank is 3 and its three *simple* roots may be denoted by α_1, α_2, α_3. Choosing α_1 and α_3 as the even roots and α_2 as the odd root, the corresponding basis elements of the Cartan subalgebra may be taken to be

$$\begin{aligned}
&\text{(i)} && \text{with } \alpha=\alpha_1,\ h_\alpha=\tfrac{1}{3}\{e_{44}-e_{66}-e_{55}+e_{77}\}\\
&\text{(ii)} && \text{with } \alpha=\alpha_2,\ h_\alpha=\tfrac{1}{3}\{e_{55}-e_{77}+e_{11}-e_{22}\}\\
&\text{(iii)} && \text{with } \alpha=\alpha_3,\ h_\alpha=-\tfrac{1}{3}\{e_{11}-e_{22}\}.
\end{aligned} \tag{8}$$

The five positive *even* roots α of osp(3/4) and their corresponding basis elements e_α may then be taken to be

$$\begin{aligned}
&\text{(i)} && \alpha=\alpha_1,\ \text{with } e_\alpha=e_{45}-e_{76}\\
&\text{(ii)} && \alpha=\alpha_1+2\alpha_2+2\alpha_3,\ \text{with } e_\alpha=e_{47}+e_{56}\\
&\text{(iii)} && \alpha=2\alpha_1+2\alpha_2+2\alpha_3,\ \text{with } e_\alpha=2e_{46}\\
&\text{(iv)} && \alpha=2\alpha_2+2\alpha_3,\ \text{with } e_\alpha=e_{57}\\
&\text{(v)} && \alpha=\alpha_3,\ \text{with } e_\alpha=e_{13}-e_{32}.
\end{aligned} \tag{9}$$

Similarly the six positive *odd* roots α of osp(3/4) and their corresponding basis elements e_α may be taken to be

$$\begin{aligned}
&\text{(i)} && \alpha=\alpha_1+\alpha_2+\alpha_3, \text{ with } e_\alpha=e_{36}-e_{43}\\
&\text{(ii)} && \alpha=\alpha_2+\alpha_3, \text{ with } e_\alpha=e_{37}-e_{53}\\
&\text{(iii)} && \alpha=\alpha_1+\alpha_2+2\alpha_3, \text{ with } e_\alpha=e_{16}-e_{42}\\
&\text{(iv)} && \alpha=\alpha_2+2\alpha_3, \text{ with } e_\alpha=e_{17}-e_{52}\\
&\text{(v)} && \alpha=\alpha_1+\alpha_2, \text{ with } e_\alpha=e_{26}-e_{41}\\
&\text{(vi)} && \alpha=\alpha_2, \text{ with } e_\alpha=e_{27}-e_{51}.
\end{aligned} \tag{10}$$

To each positive root α there corresponds a negative root $-\alpha$. All the root subspaces are one dimensional.

The basis of the osp(3/2) subalgebra of osp(3/4) may be taken to consist of all the basis elements of osp(3/4) whose fourth and sixth rows and columns consist entirely of zero matrix elements. Consequently the two-dimensional Cartan subalgebra of osp(3/2) consists of h_α with $\alpha=\alpha_2$ and α_3, the two positive even roots of osp(3/2) are $\alpha=2\alpha_2+2\alpha_3$ and α_3, and the three positive odd roots of osp(3/2) are $\alpha=\alpha_2+\alpha_3$, $\alpha_2+2\alpha_3$ and α_2. Again to each positive root α there corresponds a negative root $-\alpha$, and all the root subspaces are one dimensional.

Similarly the basis of the osp(2/2) subalgebra of this osp(3/2) superalgebra may be taken to consist of all the basis elements of osp(3/2) whose third row and column consist entirely of zero matrix elements. Consequently the two-dimensional Cartan subalgebra of osp(2/2) consists of h_α with $\alpha=\alpha_2$ and α_3, the one positive even root of osp(2/2) is $2\alpha_2+2\alpha_3$ and the two positive odd roots of osp(2/2) are $\alpha_2+2\alpha_3$ and α_2. As before, to each positive root α there corresponds a negative root $-\alpha$, and all the root subspaces are one dimensional. Clearly the complement of the subalgebra osp(2/2) in osp(3/2), which will be denoted by $\text{osp}(2/2)_{\text{comp}}$, is the four-dimensional subspace of osp(3/2) (and also of osp(3/4)) that has as its basis the two even elements e_α with $\alpha=\pm\alpha_3$ and the two odd elements e_α with $\alpha=\pm(\alpha_2+\alpha_3)$.

The Heisenberg superalgebra sh(2/2) (Beckers and Hussin 1986) consists of an identity I, two other even basis elements P_+ and P_- and two odd basis elements T_+ and T_-, the only non-zero commutation ($[.,.]_-$) and anticommutation ($[.,.]_+$) relations being assumed to be

$$[P_-, P_+]_-=2\omega I \qquad [T_-, T_+]_+=I. \tag{11}$$

Such a Heisenberg superalgebra sh(2/2) can also be embedded in osp(3/4) by making the following identifications:

$$\begin{aligned}
&\text{(i)} && I=e_\alpha, \text{ with } \alpha=2\alpha_1+2\alpha_2+2\alpha_3\\
&\text{(ii)} && P_+=(2\omega)^{1/2}e_\alpha, \text{ with } \alpha=\alpha_1+2\alpha_2+2\alpha_3\\
&\text{(iii)} && P_-=(2\omega)^{1/2}e_\alpha, \text{ with } \alpha=\alpha_1\\
&\text{(iv)} && T_+=e_\alpha, \text{ with } \alpha=\alpha_1+\alpha_2+2\alpha_3\\
&\text{(v)} && T_-=-e_\alpha, \text{ with } \alpha=\alpha_1+\alpha_2.
\end{aligned} \tag{12}$$

Apart from the fairly trivial numerical factors, all the commutation and anticommutation relations of sh(2/2) follow from the well known theorem in the theory of simple Lie superalgebras (cf Kac 1977a, b, Scheunert 1979, Cornwell 1989) that states that if

α and β are roots of a simple Lie superalgebra $\tilde{\mathscr{L}}_s$ and if $e_\alpha \in \tilde{\mathscr{L}}_{s\alpha}$ and $e_\beta \in \tilde{\mathscr{L}}_{s\beta}$ then $[e_\alpha, e_\beta] \in \tilde{\mathscr{L}}_{s(\alpha+\beta)}$ if $\alpha+\beta$ is a root of $\tilde{\mathscr{L}}_s$ and $[e_\alpha, e_\beta]=0$ if $\alpha+\beta$ is not a root of $\tilde{\mathscr{L}}_s$. Here evidently $[e_\alpha, e_\beta]$ indicates the commutator if e_α and (or) e_β are (is) even and represents the anticommutator if both e_α and e_β are odd. Also $\tilde{\mathscr{L}}_{s\alpha}$, $\tilde{\mathscr{L}}_{s\beta}$ and $\tilde{\mathscr{L}}_{s(\alpha+\beta)}$ denote the root subspaces of the roots α, β and $\alpha+\beta$.

The identifications (12) are then consequences of the following observations.

(*a*) The even root $2\alpha_1+2\alpha_2+2\alpha_3$ can be written *both* as the sum of the two even roots $\alpha_1+2\alpha_2+2\alpha_3$ and α_1 *and* as the sum of the two odd roots $\alpha_1+\alpha_2+2\alpha_3$ and $\alpha_1+\alpha_2$.

(*b*) If α is any of these five roots of osp(3/4) then 2α is *not* a root of osp(3/4).

(*c*) If α and β are any pair of these five roots of osp(3/4) (*except* for the two pairs $\alpha=\alpha_1+2\alpha_2+2\alpha_3$, $\beta=\alpha_1$ and $\alpha=\alpha_1+\alpha_2+2\alpha_3$, $\beta=\alpha_1+\alpha_2$ of (*a*)) then $\alpha+\beta$ is *not* a root of osp(3/4).

With this information let us now explore the relationships between these subalgebras of osp(3/4).

(A) The subspace spanned by the basis elements of osp(2/2) and sh(2/2) together form a *subalgebra* of osp(3/4) which has the *semi-direct sum structure* $\{\mathrm{osp}(2/2) \oplus \mathrm{sh}(2/2)\}$. This follows from the theorem mentioned above and the fact that the sum $\alpha+\beta$ of every root α of osp(2/2) with every root β corresponding to an element e_β of sh(2/2) is either not a root of osp(3/4) or is a root corresponding to an element of sh(2/2).

(B) If a is any of the elements P_+, P_-, T_+ or T_- of sh(2/2), then, with the choice $\gamma=-(\alpha_1+\alpha_2+\alpha_3)$, the *odd* element e_γ has the property that

$$[a, e_\gamma] \in \mathrm{osp}(2/2)_{\mathrm{comp}} \tag{13}$$

and every basis element of $\mathrm{osp}(2/2)_{\mathrm{comp}}$ appears in this way. That is, there is a one-to-one correspondence between the elements of the Heisenberg superalgebra sh(2/2) (apart from its identity) and the elements of $\mathrm{osp}(2/2)_{\mathrm{comp}}$. As $[P_\pm, e_\gamma]_-$ are *odd* and $[T_\pm, e_\gamma]_+$ are *even*, this explains the 'character reversal' phenomenon observed by Beckers *et al* (1988b). These results are consequences of the theorem stated above and the observation that if α is any of the roots associated with P_+, P_-, T_+ or T_- and $\gamma=-(\alpha_1+\alpha_2+\alpha_3)$ then $e_{\alpha+\gamma}$ is a member of $\mathrm{osp}(2/2)_{\mathrm{comp}}$.

It is worth mentioning that the following two further results on the structure of osp(3/2) that were noted by Beckers *et al* (1988b) in the context of the supersymmetric theory of harmonic oscillators also appear naturally in this canonical Lie superalgebra framework.

(C) If a is any element of osp(2/2) and b is any element of $\mathrm{osp}(2/2)_{\mathrm{comp}}$, then

$$[a, b] \in \mathrm{osp}(2/2)_{\mathrm{comp}} \tag{14}$$

and every element of $\mathrm{osp}(2/2)_{\mathrm{comp}}$ appears in this way. This can be written more concisely as

$$[\mathrm{osp}(2/2), \mathrm{osp}(2/2)_{\mathrm{comp}}] = \mathrm{osp}(2/2)_{\mathrm{comp}}. \tag{15}$$

(D) If a and b are any two elements of $\mathrm{osp}(2/2)_{\mathrm{comp}}$, then

$$[a, b] \in \mathrm{osp}(2/2) \tag{16}$$

and every element of osp(2/2) appears in this way. Again this can be written more concisely as

$$[\mathrm{osp}(2/2)_{\mathrm{comp}}, \mathrm{osp}(2/2)_{\mathrm{comp}}] = \mathrm{osp}(2/2). \tag{17}$$

Both of these results are immediate consequences of the root theorem stated above and the root structures of osp(2/2) and osp$(2/2)_{\text{comp}}$.

All the above properties make thus clear the correspondence (3) between the superalgebras osp(2/2)$\oplus$sh(2/2) and osp(3/2). Let us also point out that, for a one-dimensional harmonic oscillator of angular frequency ω, the physical content of the corresponding superalgebras is very easy to recover in connection with the previous works (Beckers *et al* 1988a, b). For example, the generators of the two-dimensional Cartan subalgebra of osp(3/2) or osp(2/2) can be identified with the bosonic and fermionic Hamiltonians given by

$$H_{\mathrm{B}} = T_{11} = \tfrac{1}{2}(p^2 + \omega^2 x^2) \qquad H_{\mathrm{F}} = Y = \tfrac{1}{2}\omega\sigma_3$$

while the odd roots $\pm\alpha_2$ lead to the supercharges $Q_{\pm}$, etc.

As a final comment let us mention that these $n = 1$ considerations can be extended to the n-dimensional context pointing out specific orthosymplectic chains of physically meaningful superalgebras.

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